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On the Determination of the Free Electron Distribution of an Ionized Gas

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Abstract

A standard simplified model of the problem of determining the free electron distribution in an ionized gas by radio sounding experiments is discussed, and a relation between the elements of the corresponding scattering matrix is derived. Under the conditions assumed, a closed form expression for the charge distribution in terms of a reflection coefficient which is a rational function of frequency is obtained.

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1. Introduction

This paper will deal with certain mathematical aspects of the radio sounding method for investigating the negative free charge density of an ionized gas such as the ionosphere.

A typical radio sounding experiment is conducted by sending a pulse modulated electromagnetic wave into the gas. The time delay before the reflected pulse returns is recorded for every value of the carrier frequency over a continuous band of frequencies, providing data which then become available for studying the effective index of refraction of the ionized medium.[1]

The index of refraction presented to a propagating electromagnetic wave by an ionized gas is determined for all practical purposes by the free electrons in the gas. The square of the index of refraction n is given approximately by the formula[1]

$$n^2 = 1 - \frac{Ne^2}{m\omega^2 \varepsilon_0} , \tag{1.1}$$

where the quantities appearing in (1.1) are defined as follows:

N is the density of the free electrons

e is the charge of a single electron

m is the mass of a single electron

ω is the angular frequency of the propagating wave

ε_0 is the dielectric constant of free space.

The expression (1.1) for n^2 is valid only to the extent that one can neglect losses due to collisions of the electrons. If the mean free path of the free electrons is very long compared to the wavelength of the propagating wave, the collisional frequency and thus the losses due to absorption of electromagnetic energy by the gas will be small.

It will be assumed here that the electron density in (1.1) is a function of one variable x, the height, i.e., that the electron distribution is uniform in any plane orthogonal to the direction of the x axis in a Cartesian coordinate system imbedded in the gas. Further, the coordinate system will be so chosen that the ionized medium occurs only in the region where x is positive, so that

$$N = 0 \quad \text{when} \quad x < 0 \,. \tag{1.2}$$

Given all the assumptions presented so far, a steady state periodic electric field u(x,k) which propagates in the ionized medium must satisfy a differential equation of the form

$$\frac{d^2u(x,k)}{dx} + \left\{k^2 - V(x)\right\}u(x,k) = 0 \,, \qquad (V(x) = 0 \text{ when } x < 0) \,, \tag{1.3}$$

where k is the wave number and V(x) is proportional to the electron density at x but is independent of k.

The time delay data of a radio sounding experiment can be used to find the phase of the complex reflection coefficient at x = 0 for every frequency within the experimental band [2], [3]. The phase of the reflection coefficient at other frequencies and its modulus are not given by the experimental data, although in order to obtain the electron distribution it is necessary to know the complete reflection coefficient at every value of the frequency [4], [5]. Therefore, it will be assumed that the reflection coefficient is given completely for all frequencies by some kind of extrapolation from the experimental data. Since these data are not sufficient to provide all the necessary information, it should be expected that only certain features of the resulting electron density function will actually be determined correctly. The question of what features of the electron density are determined correctly by means of the extrapolated reflection coefficient is an important but difficult one; no attempt will be made to settle it here. It must be observed, however, that the results derived from the extrapolated reflection coefficient are always at least as reliable as any result obtained from the given data alone.*

In the sections which follow, a method will be given for reconstructing the electron distribution from a knowledge of the complete reflection coefficient at all frequencies. It will be seen, in fact, that the result can be stated explicitly in closed form when the reflection coefficient is a rational function of the frequency.** The procedure for extrapolating to the complete reflection coefficient, starting from time delay data, will be described in a future article which will deal with the application of results presented here.

Section 2 of this paper can be considered an extension of the introduction, for nothing which is discussed there is new. Some of the general well-known properties of the scattering matrix associated with the differential equation (1.3) are described in Section 2 for convenient reference later on. Specific reference to the literature in that discussion will not be made because most of the material comes from a commonly-held body of knowledge difficult to attribute to any individual.

In Section 3 a derivation will be given of a method for reconstructing $V(x)$ in the differential equation (1.3) from the complete reflection coefficient at $x = 0$. Although the method has already been presented in other places [4], [5], the derivation given here is more physically motivated and slightly more general than the one previously used.

---

*While the standard method for reconstructing the electron distribution apparently makes use of nothing but the available time delay data (See [1], [6]), it should be pointed out that some unjustified hidden assumptions are contained in the procedure (See [2], [7].).

**It was found by Bargmann [8] that one can reconstruct the potential for the radial Schrödinger equation of quantum mechanics from a knowledge of the asymptotic phase shift of the wave function under similar circumstances.

Section 4 is a discussion of the scattering matrix elements when the reflection coefficient is a rational function of the wave number k.

Section 5 is devoted to proving a theorem connecting the asymptotic form of the wave u(x,k) at large distances with the reflection coefficient at x = 0 under the assumption that the reflection coefficient is a rational function of the wave number. At the same time an explicit closed form expression for V(x) in terms of elementary functions is derived. It will be seen also that u(x,k), itself, can be given explicitly in terms of elementary functions.

2. The Scattering Matrix

If V(x) in the differential equation (1.3) satisfies appropriate conditions, any solution u(x,k) has the asymptotic behavior

$$\begin{aligned} &\lim_{x \to +\infty} \left\{ A_+ e^{ikx} + B_+ e^{-ikx} - u(x,k) \right\} = 0 \\ (2.1)\qquad &\lim_{x \to -\infty} \left\{ A_- e^{ikx} + B_- e^{-ikx} - u(x,k) \right\} = 0 , \end{aligned}$$

where the coefficients $A_\pm$, $B_\pm$ are independent of x but may depend on k.

According to (2.1) the asymptotic forms of u(x,k) at large positive and negative values of x can be regarded as simple waves moving to the left and right in free space where the function V(x) vanishes. On the basis of the convention that the suppressed periodic time factor in the waves is $e^{-i\omega t}$ a left moving wave can be defined as one asymptotically proportional to e^{-ikx} and a right moving wave as one asymptotically proportional to e^{ikx} . In addition, an incoming wave can be defined as a right moving wave at $-\infty$ or a left moving wave at $+\infty$, while an outgoing wave can be defined as a right moving wave at $+\infty$ or a left moving wave at $-\infty$.

The set of all *incoming waves* can be regarded as a two dimensional vector space of column vectors such that the first component of each vector is the amplitude of the *right moving* part and the second component is the amplitude of the *left moving* part. Similarly, the *outgoing waves* can be represented by a two dimensional column vector space for which the first component of each vector is the amplitude of the *right moving* part and the second component is the amplitude of the *left moving* part.

Corresponding to every differential equation of the form (1.3) whose solutions have the asymptotic behavior (2.1) there exists a matrix, called the *scattering matrix*, which transforms each vector in the *incoming wave* vector space into a vector of the *outgoing wave* vector space. If the *incoming wave* vector contains the amplitudes of the incoming part of $u(x,k)$ in the asymptotic limits (2.1), the corresponding *outgoing wave* vector obtained by means of the scattering matrix will contain the amplitudes of the outgoing part. The scattering matrix has the form:

$$S = \begin{pmatrix} t(k) \,, & \rho(k) \\ r(k) \,, & \tau(k) \end{pmatrix} ; \tag{2.2}$$

the elements of S are called:

the transmission coefficient on the right : $t(k)$
the transmission coefficient on the left : $\tau(k)$
the reflection coefficient on the right : $\rho(k)$
the reflection coefficient on the left : $r(k)$

If, for example,

$$\lim_{x \to -\infty} \left\{ u(x,k) - (A_- e^{ikx} + B_- e^{-ikx}) \right\} = 0 \,,$$

and

$$\lim_{x \to +\infty} \left\{ u(x,k) - (A_+ e^{ikx} + B_+ e^{-ikx}) \right\} = 0 \,,$$

then

$$\begin{pmatrix} A_+ \\ B_- \end{pmatrix} = S \begin{pmatrix} A_- \\ B_+ \end{pmatrix}$$

or equivalently,

$$(2.3)\qquad \begin{aligned} A_+ &= A_-t(k) + B_+\rho(k) \\ B_- &= A_-r(k) + B_+\tau(k) \ . \end{aligned}$$

Several facts about the scattering matrix are well known. If $V(x)$ in (1.3) is sufficiently well-behaved, the elements of S are analytic functions of the wave number k in the complex plane, and these functions are regular on the real axis. Furthermore, under the condition on $V(x)$ in (1.3) $r(k)$ and $t(k)$ are regular in the upper half plane and on the real axis, and in the upper half plane they obey the asymptotic conditions for large k:

$$(2.4)\qquad r(k) = O(\tfrac{1}{k}),\ t(k) = 1 + O(\tfrac{1}{k}) \ .$$

The following relations hold in general for real k:

$$(2.5)\qquad r(-k)=\overline{r(k)},\ t(-k)=\overline{t(k)},\ \rho(-k)=\overline{\rho(k)},\ \tau(-k)=\overline{\tau(k)} \ ,$$

where the bar over a number indicates the complex conjugate. It is also known that S is a unitary matrix when k is real:

$$(2.6)\qquad S^*(k) = S^{-1}(k) \ ,\ k \text{ real.}$$

In addition, the reciprocity law holds:

$$(2.7)\qquad t(k) = \tau(k)$$

for all k.

Relations (2.5), (2.6) and (2.7) imply for real k:

$$(2.8)\qquad \begin{aligned} &|r(k)|^2+|t(k)|^2 = r(k)r(-k)+t(k)t(-k) = 1 \text{ (energy conservation)} \\ &\rho(k)\overline{t(k)}+\overline{r(k)}t(k) = \rho(k)t(-k)+r(-k)t(k) = 0 \text{ (phase law)} \ . \end{aligned}$$

3. The Inverse Problem

Associated with the differential equation of (1.3) was the condition

$$V(x) = 0, \quad x < 0 . \tag{3.1}$$

Because of (3.1) the asymptotic limit of $u(x,k)$ given by (2.1) for $x \longrightarrow -\infty$ is identical with $u(x,k)$ itself for negative values of x. If in (2.3)

$$A_- = 1, \quad B_+ = 0 ,$$

then

$$u(x,k) = e^{ikx} + r(k)e^{-ikx} , \quad x < 0$$

(3.2)

$$\text{and} \quad \lim_{x \longrightarrow \infty} \left\{ u(x,k) - t(k)e^{ikx} \right\} = 0 .$$

The asymptotic conditions (3.2) mean that a wave of unit amplitude is incident from the left, producing a reflected wave of amplitude $r(k)$ and a transmitted wave of amplitude $t(k)$. Assuming that $r(k)$ is given for all positive real values of k, the central problem of this paper is to find the function $V(x)$ in the differential equation (1.3).

In accordance with the physical problem described in Section 1 it is assumed that no proper eigenfunctions of (1.3) exist.* Because of this fact and (3.1), $r(k)$ is regular in the upper half plane[5].

The Fourier transform $U(x,t)$ of $u(x,k)$ is a transient wave solution of the time dependent differential equation

$$\frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial^2 U(x,t)}{\partial t^2} - V(x)\, U(x,t) = 0 , \tag{3.3}$$

where the wave velocity has been taken to be one. If $u(x,k)$ satisfies the asymptotic conditions (3.2), the Fourier transform of $u(x,k)$ exists only as as improper function; i.e., it contains a Dirac delta function in the region $x < 0$. However, if the incident continuous wave of unit amplitude is replaced by one having an integrable amplitude in the interval $(-\infty, +\infty)$, the corresponding transient solution of (3.3) is a proper function. The transient solution of (3.3) corresponding to an incident steady-state continuous wave of unit amplitude may therefore be regarded as a limiting case of well-behaved transients - - useful because of its formal simplicity.

---

*Since the charge density is all negative $V(x) \geq 0$ for all x.

The reflected transient due to the incident wave

$$U_{inc} = \delta(x-t) , \quad x < 0 , \quad t < 0 .$$

is

$$(3.4) \qquad R(x+t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{-ik(x+t)} \, dk .$$

In order to obtain (3.4) when r(k) is known for positive real k it is necessary to extend r(k) to the negative real axis by using (2.5). At the same time it can be seen that R(x+t) is real for all real x and t. The total solution U(x,t) of (3.3) in the region x < 0 is then

$$(3.5) \qquad U(x,t) = \delta(x-t) + R(x+t) , \quad x < 0 ,$$

and U(x,t) is real for all x and t.

The function R(z) has the property

$$(3.6) \qquad R(z) = 0 , \quad z < 0$$

because of (2.4) and the regularity of r(k) in the upper half plane. The physical meaning of (3.6) in connection with (3.5) is easily understood, for the reflected transient should not appear at any point x until the appropriate amount of time has elapsed after the incident transient has struck the edge of the reflecting medium at x = 0.

It is reasonable to expect that any solution U(x,t) of (3.3) can be obtained by applying a linear transformation to some solution $U_o(x,t)$ of the free space wave equation

$$(3.7) \qquad \frac{\partial^2 U_o}{\partial x^2} - \frac{\partial^2 U_o}{\partial t^2} = 0 .$$

In fact it would not be unreasonable to suppose that there exists a kernel function $K(x,z)$ such that

$$(3.8) \qquad U(x,t) = U_0(x,t) + \int_{-\infty}^{\infty} K(x,z)\, U_0(z,t)\,dz \, .$$

Now, if $U(x,t)$ is a right moving transient in the sense that

$$(3.9) \qquad U(x,t) = 0 \, , \quad x > t \, ,$$

then it would seem natural to suppose that $U(x,t)$ will depend only on $U_0(x,t)$ at those values of the space variable x which the non zero part of the traveling wave would have had time to reach at the time t. In other words, one might guess that

$$(3.10) \qquad K(x,z) = 0, \quad z > x \, .$$

Assuming (3.9) and (3.10), (3.8) becomes

$$(3.11) \qquad U(x,t) = U_0(x,t) + \int_{-\infty}^{x} K(x,z)\, U_0(z,t)\,dz \, .$$

If $U_0(x,t)$ is the particular solution of (3.7):

$$(3.12) \qquad U_0(x,t) = \delta(x-t) + R(x+t) \, ,$$

then (3.11) becomes

$$(3.13) \qquad U(x,t) = \delta(x-t) + R(x+t) + K(x,t)\eta(x-t) + \int_{-t}^{x} K(x,z)\, \eta(x-z) R(z+t)\,dz \, ,$$

where

$$\eta(z) = \begin{cases} 0 \, , & z < 0 \\ 1 \, , & z > 0 \, . \end{cases}$$

Assuming that

$$(3.14) \qquad K(x,t) = 0 \, , \quad t \leq -x \, ,$$

it follows from (3.13) that

$$U(x,t) = U_0(x,t) \, , \quad x < 0 \, .$$

It can be shown directly that there exists a function $K(x,z)$ satisfying (3.10) and (3.14) such that (3.11) holds[5]. Applying (3.14) to (3.11) one obtains

$$(3.15)\qquad U(x,t) = \begin{cases} U_0(x,t) + \int_{-x}^{x} K(x,z)\, U_0(z,t)dz\ , & x \geq 0 \\ U_0(x,t)\ , & x \leq 0\ . \end{cases}$$

After substituting from (3.15) into (3.3), using (3.7), it is easy to verify that (3.15) provides a solution $U(x,t)$ of (3.3) if $K(x,z)$ satisfies

$$(3.16)\qquad \frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial z^2} - V(x)K = 0\ , \quad z < x\ ,$$

subject to the conditions

$$(3.17)\qquad K(x,-x) = 0,\ \ 2\,\frac{dK(x,x)}{dx} = V(x)\ .^{*}$$

Since it is known from the theory of the linear second order hyperbolic partial differential equation that the Cauchy problem defined by (3.16) and (3.17)* has a solution, the existence of $K(x,z)$ satisfying the required conditions follows and thus the expression (3.15) for $U(x,t)$[5].

Now consider (3.15) with $U_0(x,t)$ given by (3.12). Assume that the corresponding $U(x,t)$ is a right moving wave; i.e.,

$$(3.18)\qquad U(x,t) = 0\ , \quad x > t\ .$$

Then (3.15) implies

$$(3.19)\qquad 0 = R(x+t) + K(x,t) + \int_{-\infty}^{x} K(x,z)R(z+t)dz\ , \quad x > t\ .$$

---

[*] Conditions (3.17) and continuity of $K(x,z)$ in the region $z \leq x$ imply

$$K(x,x) = \begin{cases} \frac{1}{2}\int_0^x V(x)dx\ , & x \geq 0 \\ 0\ , & x \leq 0 \end{cases}.$$

Since, according to (3.6), $R(z+t)$ is zero for $z<-t$ the lower limit in the integral of (3.19) can be replaced by $-t$, and it follows from (3.19) that

$$K(x,t) = 0 , \quad t < -x . \tag{3.20}$$

Equation (3.19) is a Fredholm integral equation for $K(x,t)$ considered as a function of t (with x regarded as a parameter) with the kernel $R(z+t)$ over the range $-\infty < t < x$. When $R(z+t)$ is given, e.g., by experimental data through (3.4), the integral equation (3.19) can be solved for $K(x,t)$ at every value of x. The function $V(x)$ can then be obtained from the second relation in (3.17).

The existence of a solution $K(x,t)$ of (3.19) can be established by means of a standard argument in the Fredholm theory: one simply proves the uniqueness of the solution.* At the same time the uniqueness of the solution $K(x,t)$ of (3.19) can be used to establish the fact that $K(x,t)$ satisfies (3.16) when $V(x)$ is defined by (3.17). The first relation of (3.17) follows at once from (3.19) and (3.6). The fact that $K(x,t)$ satisfies (3.16) follows without much difficulty on applying the differential operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - V(x)$$

to (3.19). The result is

$$0 = \left\{\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - V(x)K\right\} + \int_{-\infty}^{x} \left\{\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - V(x)K\right\} R(z+t)\,dz, \quad x > t . \tag{3.21}$$

Equation (3.21) is the homogeneous integral equation corresponding to (3.19) for the function

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - V(x)K .$$

---

*See Section 4 and [5] .

If the solution of (3.19) is unique, the solution of (3.21) must vanish:

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - V(x)K = 0$$

as required by (3.16).

4. The Case in Which r(k) is a Rational Function

If the reflection coefficient r(k) is known for all real values of k, or even for just the positive real values because of (2.5), the function V(x) and any solution u(x,k) of (1.3) can be determined as indicated in Section 3. In principle, it is then possible to find the other elements of the scattering matrix in terms of r(k). As a practical matter, however, the relationship thus exhibited between the elements of the scattering matrix is quite complicated and indirect. On the other hand, if the given r(k) is a rational function of k, an explicit relation between r(k) and t(k) or between r(k) and $\rho(k)$ can be found as well as a closed form expression for V(x). These relations between the scattering matrix elements will now be discussed.

Consider the general form which r(k) can take if it is a rational function of k:

$$(4.1) \qquad r(k) = \frac{r_o \prod_{i=1}^{m} (k-\mu_i)}{\prod_{i=1}^{n} (k-\lambda_i)},$$

where r_o, the μ_i and the λ_i are complex constants, and the integers m and n satisfy the inequality

$$(4.2) \qquad m \leq n + 2 .$$

The condition (4.2) is necessary if a piecewise continuous V(x) is desired, i.e., if r(k) is not $o(\frac{1}{k})$ for large k, V(x) will contain a part which is improper, e.g., a Dirac delta function. This can be seen by examining the method of Section 3. The complex constants r_o, λ_i, μ_i must be such that the reality condition on r(k) given by (2.5) is satisfied. Also, to comply with condition (3.1) which depends on (3.6) and to prevent the occurrence of proper eigenfunctions[5] r(k) must be regular in the upper half of the complex k plane. In other words, the constants λ_i are all in the lower half plane:

(4.3) $$\operatorname{Im} \lambda_i < 0, \; i = 1,2, \ldots, n \; .$$

Finally, it is convenient to impose the condition

(4.4) $$|r(k)| < 1, \text{ for all real } k \; .$$

Condition (4.4) is slightly stronger than the necessary one

(4.5) $$|r(k)| \leq 1, \text{ for all real } k,$$

implied by (2.8), but it is not essentially more restrictive since the r(k) satisfying (4.5) can be multiplied by a parameter ε such that

$$0 < \varepsilon < 1$$

and the correct result obtained as a limiting case when ε approaches one. In the same spirit, it will be assumed that the numbers $\varkappa_i$, which will be introduced later as zeros of 1-r(k)r(-k) in the upper half plane, are all distinct, the case of multiple zeros being treated as a limiting case in which two or more zeros coalesce.

If some a priori assumptions are made about t(k), by a standard argument the connection between t(k) and r(k) can be given even for more general reflection coefficients r(k). Suppose, in fact, it is assumed that t(k) has no zeros or singularities in the upper half of the complex k plane. For real k, by (2.8) and (2.5):

(4.6) $$|t(k)| = \sqrt{1-|r(k)|^2} = \sqrt{1-r(k)r(-k)} \; .$$

The function

$$\log t(k) = \log|t(k)| + i\,\theta(k)\,, \tag{4.7}$$

k real, is analytic on the real axis and in the upper half plane according to the assumptions made about t(k). The phase $\theta(k)$ of t(k) is an odd function of k and the modulus $|t(k)|$ is an even function of k when k is real according to (2.5). Also by (2.4)

$$\lim_{k \longrightarrow \infty} \log t(k) = 0\,, \tag{4.8}$$

if the limit is approached on the real axis or in the upper half plane. By Cauchy's integral theorem, integrating over an infinite semi-circle in the upper half plane and over the real axis, and by (4.8)

$$\log t(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log t(\sigma)}{\sigma - k}\,d\sigma\,, \tag{4.9}$$

k in the upper half plane. Moreover, since $\bar{k}$ is outside the closed contour

$$0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log t(\sigma)}{\sigma - \bar{k}}\,d\sigma\,, \tag{4.10}$$

k in the upper half plane. On taking the complex conjugate of both sides, (4.10) becomes

$$0 = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\overline{\log t(\sigma)}}{\sigma - k}\,d\sigma\,, \tag{4.11}$$

k in the upper half plane. Subtracting (4.11) from (4.9) gives the result:

$$\log t(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\left\{1 - r(\sigma)r(-\sigma)\right\}}{\sigma - k}\,d\sigma\,, \tag{4.12}$$

k in the upper half plane, where (4.7) and (4.6) have been used. Equation (4.12) provides the general relation between t(k) and r(k) when t(k) is regular and has no zeros in the upper half plane. If these conditions are allowed to be violated, the zeros and singularities of t(k) must be known in order to specify a relationship between r(k) and t(k) by means of an argument such as the one just used, based entirely on the general properties of the scattering matrix elements given in Section 2. Apparently, if this information were lacking, it would be necessary in some way to include more of the relationship of the scattering matrix to the differential equation (1.3).

If (4.12) is applied to the r(k) given by (4.1), an explicit formula for t(k) can be obtained. The evaluation of the integral in (4.12) can be carried out by considering first

$$\frac{d}{dk}\log t(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\{1-r(\sigma)r(-\sigma)\}}{(\sigma-k)^2}\,d\sigma , \tag{4.13}$$

k in the upper half plane. After an integration by parts (4.13) becomes

$$\frac{d}{dk}\log t(k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\frac{d}{d\sigma}\{r(\sigma)r(-\sigma)\}}{(\sigma-k)\{1-r(\sigma)r(-\sigma)\}}\,d\sigma , \tag{4.14}$$

k in the upper half plane. The integrand of the integral on the right of (4.14) is now a rational function of σ and can, therefore, be evaluated by the method of residues. The result is a rational function of k equal to the derivative of log t(k). The integration of that function is easily carried out and the constant of integration determined by the condition (4.8). The result is

$$t(k) = \frac{\prod_{i=1}^{n}(k+\varkappa_i)}{\prod_{i=1}^{n}(k-\lambda_i)} , \tag{4.15}$$

where the n complex constants $\varkappa_i$ are the n roots in the upper half plane of the equation

$$1-r(k)r(-k) = 0 . \tag{4.16}$$

That there are exactly n roots of (4.16) in the upper half plane is clear from the invariance of (4.16) when k is replaced by -k and from condition (4.4).

The numbers x_i are assumed to be distinct, i.e., it is assumed that there are no multiple roots of the equation (4.16). If this is not so, r(k) can be modified slightly so that the multiple roots of (4.16) split up into simple roots, and one can consider the original case as a limiting form in which two or more roots of (4.16) coalesce. It is obvious that all conditions on t(k) are satisfied by (4.15) in any case, even if one ignores its derivation.

The phase law of (2.8) provides the means for obtaining an expression for the function $\rho(k)$ once t(k) is assumed to be given by (4.15):

$$\rho(k) = \frac{(-1)^{m-n+1} r_o \prod_{i=1}^{m} (k+\mu_i) \prod_{j=1}^{n} (k+x_j)}{\prod_{i=1}^{n} (k-\lambda_i) \prod_{j=1}^{n} (k-x_j)} . \tag{4.17}$$

Notice that $\rho(k)$, unlike r(k), has poles in the upper half plane: the x_i. Since no proper eigenfunctions can exist it is clear from Section 3 that the function V(x) does not vanish identically for all values of x larger than some finite number; i.e., V(x) must be different from zero in the positive x direction at arbitrarily large values of x.

Unfortunately the results of (4.15) and (4.17) cannot be trusted on the basis of the argument given for their derivation because of the *a priori* assumptions made on the t(k) that it has no zeros or singularities in the upper half plane. Instead of attempting to show somehow that these assumptions are justified by properties of the differential equation (1.3) there will be given an independent derivation which will at the same time provide a closed form expression for V(x).

The relations (4.15) and (4.17) are demonstrated by starting with $\rho(k)$ as given in (4.17) and treating it as the reflection coefficient on the left for the differential equation (1.3) with x replaced by -x. The reflection coefficient on the right in this case will be r(k). Because V(-x) extends to infinity on the left the solution u(-x,k), which can be obtained from $\rho(k)$ by the method of Section 3, will be found to have a comparatively simple form. If, instead, r(k) were used to obtain the solution u(x,k) corresponding to V(x), which vanishes for negative values of x, it would be found that u(x,k) is much more complicated in form. The demonstration can be carried out by showing that u(-x,k) and its derivative at x = 0 match the assumed form of u(-x,k) and its derivative given in terms of r(k) at positive values of x.

Temporarily it is convenient to define:

$$\hat{\rho}(k) = \frac{(-1)^{m-n+1}\, r_o \prod_{i=1}^{m} (k+\mu_i) \prod_{j=1}^{n} (k+\varkappa_j)}{\prod_{i=1}^{n} (k-\lambda_i) \prod_{j=1}^{n} (k-\varkappa_j)} \tag{4.18}$$

and

$$\hat{t}(k) = \frac{\prod_{i=1}^{n} (k+\varkappa_i)}{\prod_{j=1}^{n} (k-\lambda_i)} \; . \tag{4.19}$$

__Lemma:__ for all real k

$$\begin{gathered} \hat{\rho}(-k) = \overline{\hat{\rho}(k)}, \quad \hat{t}(-k) = \overline{\hat{t}(k)} \; , \\ |r(k)| = |\hat{\rho}(k)|, \; |r(k)|^2 + |\hat{t}(k)|^2 = 1 \; , \\ \bar{r}(k)\, \hat{t}(k) + \hat{\rho}(k)\, \overline{\hat{t}(k)} = 0 \; . \end{gathered} \tag{4.20}$$

__Proof:__ Consider

$$P_1(k) = \hat{t}(k)\hat{t}(-k) \prod_{i=1}^{n}(k-\lambda_i) \prod_{j=1}^{n}(-k-\lambda_j) \tag{4.21}$$

and

$$(4.22) \qquad P_2(k) = \left\{1-r(k)r(-k)\right\} \prod_{i=1}^{n} (k-\lambda_i) \prod_{j=1}^{n} (-k-\lambda_j) \; .$$

From (4.1) and (4.19) it can be seen that $P_1(k)$ and $P_2(k)$ are each polynomials in k of order 2n. The roots of $P_1(k)$ are the 2n numbers $\pm x_i$, i=1, ..., n . By the definition of the x_i in connection with (4.16) and the fact that replacing k by -k in (4.16) leaves the equation unchanged it is obvious that the numbers $\pm x_i$, i=1, ..., n are also the roots of $P_2(k)$. After dividing (4.21) and (4.22) by $\prod_{i=1}^{n} (k-\lambda_i) \prod_{j=1}^{n} (-k-\lambda_j)$ and allowing k to approach ∞ it is found that each of the expressions approaches one. The conclusion is

$$(4.23) \qquad \hat{t}(k)\hat{t}(-k) = 1 - r(k)r(-k) \; .$$

It follows from (4.23), (4.4) and (2.5) that $\hat{t}(k)\hat{t}(-k)$ is real and positive for real

By (4.1) and (2.5), for real k,

$$(4.24) \qquad r(-k) = \frac{\bar{r}_o \prod_{i=1}^{n} (k-\bar{\mu}_i)}{\prod_{i=1}^{n} (k-\bar{\lambda}_i)} \; .$$

Analytic continuation implies the validity of (4.24) for all k. From (4.1), (4.23) and (4.24) it can be seen that the 2n poles of $\hat{t}(k)\hat{t}(-k)$ are λ_i, i=1, ..., n and $\bar{\lambda}_i$, i=1, ..., n. Each λ_i is distinct from all the $\bar{\lambda}_i$ since the λ_i are given in the lower half plane. It can then be seen that the 2n roots of (4.16) consist of the 2n numbers, $\pm x_i$, i=1, ..., n as well as the 2n numbers $\pm \overline{x_i}$, i=1, ..., n. Of course (4.16) has at most 2n distinct roots altogether; by the assumption of this article, in fact, it has exactly 2n distinct roots. By (4.23), therefore, $\hat{t}(k)\hat{t}(-k)$ has as zeros the $\pm x_i$, i=1, ..., n as well as the $\pm \bar{x}_i$, i=1, ..., n. Thus, the rational function

$$\hat{t}(k)\overline{\hat{t}(k)} = \frac{\prod_{i=1}^{n}(k+\kappa_i)\prod_{j=1}^{n}(k+\bar{\kappa}_j)}{\prod_{i=1}^{n}(k-\lambda_i)\prod_{j=1}^{n}(k-\bar{\lambda}_j)}$$

has the same poles and zeros as $\hat{t}(k)\hat{t}(-k)$, and both expressions approach one as k approaches ∞. It follows that

$$\hat{t}(-k) = \overline{\hat{t}(k)},\ k \text{ real.} \tag{4.25}$$

From (4.18) and (4.19):

$$\hat{\rho}(k) = \frac{-r(-k)\hat{t}(k)}{\hat{t}(-k)} \tag{4.26}$$

The relations (4.20) of the Lemma follow at once from (4.25), (4.26) and (4.23).

The arguments just presented concerning the zeros of $\hat{t}(k)\hat{t}(-k)$ demonstrate at the same time that the n zeros of $\hat{t}(-k)$ are the κ_i, i=1, ..., n and the $-\bar{\kappa}_i$, i=1, ..., n. It follows from (4.26) that the n poles of $\hat{\rho}(k)$ in the upper half plane are the κ_i, i=1, ..., n and the $-\bar{\kappa}_i$, i=1, ..., n, since the λ_i are all given in the lower half plane. By the Mittag-Leffler theorem the function $\hat{\rho}(k)$ can be written in the form

$$\hat{\rho}(k) = \sum_{\nu=1}^{n}\frac{\rho_\nu}{k-\lambda_\nu} + \sum_{\nu=1}^{n}\frac{\rho_{n+\nu}}{k-\kappa_\nu}\,. \tag{4.27}$$

By considering the analytic continuation of $\overline{\hat{\rho}(k)}$, k real, and comparing with $\hat{\rho}(-k)$, using (4.20), it is observed that (4.27) can be expressed as the sum of pairs of terms of the form

$$\frac{\rho_i}{k-\alpha_i} + \frac{\rho_j}{k+\alpha_i} = \frac{-\overline{\rho_i}}{k+\alpha_i} - \frac{\overline{\rho_j}}{k-\alpha_i},\ k \text{ real}.$$

It follows at once then that

$$\rho_j = -\overline{\rho_i}\,. \tag{4.28}$$

5. V(x) When r(k) is a Rational Function

Consider the problem of determining V(x) in (1.3) when the function $\hat{\rho}(k)$ given by (4.18) is the reflection coefficient at x = 0.* The kernel of the integral equation (3.19) is

$$(5.1)\quad R(x+t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\rho}(k)e^{-ik(z+t)}dk = i\sum_{\nu=1}^{n} \rho_\nu e^{-i\kappa_\nu(z+t)}\ ,\quad z+t < 0\ ,$$

where the ρ_ν are the residues of $\hat{\rho}(k)$ at its poles in the upper half plane. From the lemma of Section 4 it can be concluded that R(z+t) is real for real values of z and t.

To solve the integral equation (3.19) it is convenient to assume that when x is negative K(x,t) has the form

$$(5.2)\qquad K(x,t) = \sum_{\nu=1}^{n} f_\nu(x)e^{-i\kappa_\nu t}\ .$$

With this assumption the integral equation (3.19) takes the form

$$(5.3)\qquad -\rho_\nu \sum_{\mu=1}^{n} \frac{e^{-i(\kappa_\nu+\kappa_\mu)x}}{\kappa_\nu+\kappa_\mu} f_\mu(x) + f_\nu(x) + i\rho_\nu e^{-i\kappa_\nu x} = 0,\ \nu = 1, \ldots, n,$$

which holds when x < 0.

The system (5.3) has a unique solution for the $f_\nu(x)$. This fact can be proved in exactly the same way that the uniqueness of the solution of the integral equation (3.19) can be proved[5].

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*It will be assumed throughout that the poles of $\hat{\rho}(k)$ in the upper half plane, the κ_i, are distinct. If there are multiple poles, the reflection coefficient is to be replaced by one for which the κ_i are distinct, the poles which were formerly multiple being replaced by simple poles very near each other. The case in question then becomes a limiting case in which certain simple poles coalesce to form multiple poles.

Multiply the homogeneous system of algebraic equations corresponding to (5.3) by $e^{-i\kappa_\nu t}$ and sum over ν:

$$\sum_{\nu,\mu=1}^{n} \frac{\rho_\nu e^{-i(\kappa_\nu+\kappa_\mu)x}}{\kappa_\nu+\kappa_\mu}\, e^{-i\kappa_\nu t}\, f^o_\mu(x) + \sum_{\nu=1}^{n} e^{-i\kappa_\nu t}\, f^o_\nu(x) = 0, \quad t < x < 0 . \tag{5.4}$$

Equation (5.4) can be written

$$\begin{aligned} &\int_{-\infty}^{x} dz \left\{ \sum_{\nu=1}^{n} i\rho_\nu e^{-i\kappa_\nu(t+z)} \sum_{\mu=1}^{n} e^{-i\kappa_\mu z}\, f^o_\mu(x) \right\} + \sum_{\nu=1}^{n} e^{-i\kappa_\nu t}\, f^o_\nu(x) \\ &= \int_{-\infty}^{x} dz \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}(k) e^{-ik(z+t)} dk \sum_{\mu=1}^{n} e^{-i\kappa_\mu z}\, f^o_\mu(x) \right\} + \sum_{\nu=1}^{n} e^{-i\kappa_\nu t}\, f^o_\nu(x) = 0 , \end{aligned} \tag{5.5}$$

for $x + t < 0$, $t < x$. On setting

$$U^o(x,t) = \sum_{\nu=1}^{n} e^{-i\kappa_\nu t}\, f^o_\nu(x) , \tag{5.6}$$

equation (5.5) becomes

$$\int_{-\infty}^{x} R(t+z) U^o(x,z) dz + U^o(x,t) = 0 , \quad x + t < 0,\ t < x . \tag{5.7}$$

Equation (5.7) can be written

$$\lim_{N \longrightarrow \infty} \int_{-\infty}^{x} \left\{ \frac{1}{2\pi} \int_{-N}^{N} \hat{\rho}(k) e^{-ik(z+t)} dk + \frac{1}{2\pi} \int_{-N}^{N} e^{-ik(z-t)} dk \right\} U^o(x,z) dz = 0, \tag{5.8}$$

$$x + t < 0,\ t < x ,$$

since $U^o(x,t)$ is obviously absolutely integrable in the interval $-\infty$ to x .

Using the lemma of Section 4 it can be verified in a straightforward way that

$$\int_{-N}^{N}\left\{\hat{\rho}(k)e^{-ik(z+t)} + e^{-ik(z-t)}\right\}dk \tag{5.9}$$

$$= \int_{0}^{N}\left\{1-|\hat{\rho}(k)|^2\right\}e^{ik(t-z)}dk + \int_{-N}^{0}\left\{e^{ikt} + \hat{\rho}(k)e^{-ikt}\right\}\overline{\left[e^{ikz} + \hat{\rho}(k)e^{-ikz}\right]}dk \, .$$

After substituting (5.9) into (5.8), multiplying by $\overline{U^o(x,t)}$ and integrating over t from $-\infty$ to x, there results:

$$\lim_{N \to \infty}\int_{0}^{N}\left\{1-|\hat{\rho}(k)|^2\right\}\left\{\overline{\int_{-\infty}^{x} U^o(x,t)e^{-ikt}dt}\right\}\left\{\int_{-\infty}^{x} U^o(x,z)e^{-ikz}dz\right\}dk \tag{5.10}$$

$$+ \lim_{N \to \infty}\int_{-N}^{0}\left\{\int_{-\infty}^{x}\left[e^{ikt}+\hat{\rho}(k)e^{-ikt}\right]\overline{U^o(x,t)}dt\right\}\left\{\overline{\int_{-\infty}^{x}\left[e^{ikz}+\hat{\rho}(k)e^{-ikz}\right]\overline{U^o(x,z)}dz}\right\}dk = 0 \, ,$$

$$x < 0 \, .$$

According to (4.20), the lemma of Section 4, conservation of energy for $\hat{\rho}(k)$ and $\hat{t}(k)$ holds when k is real; i.e., $|\hat{\rho}| \leq 1$ when k is real. It follows that each term on the left of (5.10) is positive for any function $\int_{-\infty}^{x} U^o(x,t)e^{-ikt}dt$ unless it is identically zero when $x < 0$, and so the equation (5.10) implies that $U^o(x,t) = 0$ if $x + t < 0$. From (5.6), then

$$f_\nu^o(x) = 0 \, , \qquad x < 0 \, . \tag{5.11}$$

Thus, the only solution of the homogeneous system (5.4) is the trivial solution $f_\nu^o \equiv 0$, and, therefore, the system (5.3) has a unique solution.

For a reflection coefficient r(k) given by (4.1) satisfying the general conditions of Section 2 as well as the special conditions (4.2), (4.3) and (4.4) relations for the other elements t(k) and $\rho(k)$ of the scattering matrix can be given. In fact,

Theorem 1: The functions $\hat{\rho}(k)$ given by (4.18) and $\hat{t}(k)$ given by (4.19) are the scattering matrix elements $\rho(k)$ and t(k) associated with r(k) given by (4.1).

Proof: Suppose -x is substituted for x in the differential equation (1.3), so that V(-x) replaces V(x) and u(-x,k) replaces u(x,k). Set

$$W(x) = V(-x), \quad v(x,k) = u(-x,k) \ . \tag{5.12}$$

The reflection coefficient on the left, r(k), corresponding to V(x) becomes the reflection coefficient on the right corresponding to W(x). The reflection coefficient on the left corresponding to W(x) is $\rho(k)$, and the transmission coefficient on either side is t(k).

Let v(x,k) be the solution which on the right is given by

$$v(x,k) = e^{-ikx} + r(k)e^{ikx} \ , \quad x > 0 \ , \tag{5.13}$$

and on the left by

$$\lim_{x \to -\infty} \left\{ v(x,k) - t(k)e^{-ikx} \right\} = 0 \ . \tag{5.14}$$

Let $\hat{v}(x,k)$ be the solution which is determined on the left by

$$\lim_{x \to -\infty} \left\{ \hat{v}(x,k) - \hat{t}(k)e^{-ikx} \right\} = 0 \ . \tag{5.15}$$

To prove the theorem it is only necessary to prove that

$$\hat{v}(x,k) = e^{-ikx} + r(k)e^{ikx} \ , \quad x > 0 \ , \tag{5.16}$$

or equivalently that

$$(5.17) \qquad \hat{v}(o,k) = 1+r(k) \text{ and } \left.\frac{d\hat{v}}{dx}\right|_{x=0} = ik\left\{ r(k)-1 \right\} .$$

From (5.15), by taking the Fourier transform of (3.11) and substituting (5.2), the following expression for $\hat{v}(x,k)$ is obtained:

$$\hat{v}(x,k) = \hat{t}(k)e^{-ikx} + i\hat{t}(k)e^{-ikx} \sum_{\nu=1}^{n} \frac{f_\nu(x)e^{-ix_\nu x}}{x_\nu+k} \quad , \quad x < 0$$

The relations (5.17) can now be written

$$(5.19) \qquad \hat{t}(k) + i\hat{t}(k) \sum_{\nu=1}^{n} \frac{f_\nu(0)}{x_\nu+k} = 1 + r(k)$$

and

$$(5.20) \qquad -ik\hat{t}(k)+\hat{t}(k) \sum_{\nu=1}^{n} f_\nu(0) + i\hat{t}(k) \sum_{\nu=1}^{n} \frac{f'_\nu(0)}{x_\nu+k} = ik\left\{ r(k)-1\right\} .$$

In order to demonstrate (5.19) it is convenient to define the complex numbers g_ν by the equation

$$(5.21) \qquad \sum_{\nu=1}^{n} \frac{g_\nu}{k+x_\nu} = -i\left\{\frac{r(k)}{\hat{t}(k)} + \frac{1}{\hat{t}(k)} - 1\right\} .$$

The expression on the left of (5.21) is a legitimate representation of the rational function of k on the right by the Mittag-Leffler theorem, and the coefficients g_ν are given by

$$(5.22) \quad g_\nu = -i \lim_{k \to -x_\nu} (k+x_\nu)\left\{\frac{r(k)}{\hat{t}(k)} + \frac{1}{\hat{t}(k)} - 1\right\} = -i\left\{\frac{r(-x_\nu)}{\hat{t}_\nu} + \frac{1}{\hat{t}_\nu}\right\}, \ \nu=1, \ \ldots, \ n$$

where the residue of $\frac{1}{\hat{t}(k)}$ at the pole $-\varkappa_\nu$ has been designated by $\frac{1}{\hat{t}_{-\nu}}$. From (5.3)

$$f_\nu(0) = \rho_\nu\left\{\sum_{\mu=1}^{n} \frac{f_\mu(0)}{\varkappa_\nu+\varkappa_\mu} - i\right\}, \quad \nu = 1, \ldots, n\,. \tag{5.23}$$

The numbers g_ν of (5.22) will satisfy the system of equations (5.23) when substituted for the numbers $f_\nu(0)$. To see this, consider

$$\rho_\nu\left\{\sum_{\mu=1}^{n} \frac{g_\mu}{\varkappa_\nu+\varkappa_\mu} - i\right\} = -i\rho_\nu\left\{\frac{r(\varkappa_\nu)}{\hat{t}(\varkappa_\nu)} + \frac{1}{\hat{t}(\varkappa_\nu)}\right\}, \quad \nu = 1, \ldots, n\,, \tag{5.24}$$

by (5.21). The right side of (5.24) can be written

$$\begin{aligned}
&\lim_{k \to \varkappa_\nu} - i(k-\varkappa_\nu)\hat{\rho}(k)\left\{\frac{r(k)}{\hat{t}(k)} + \frac{1}{\hat{t}(k)}\right\} \\
&= \lim_{k \to \varkappa_\nu} i(k-\varkappa_\nu)\left\{\frac{1}{\hat{t}(-k)} - \hat{t}(k) + \frac{r(-k)}{\hat{t}(-k)}\right\} \\
&= -i\left\{\frac{r(-\varkappa_\nu)}{\hat{t}_{-\nu}} + \frac{1}{\hat{t}_{-\nu}}\right\}, \quad \nu = 1, \ldots, n\,,
\end{aligned} \tag{5.25}$$

making use of the lemma relations (4.20). From (5.22), (5.24) and (5.25):

$$g_\nu = \rho_\nu\left\{\sum_{\mu=1}^{n} \frac{g_\mu}{\varkappa_\nu+\varkappa_\mu} - i\right\}, \quad \nu = 1, \ldots, n; \tag{5.26}$$

i.e., the g_ν satisfy the system of equations (5.23).

Since the system (5.23) must have a unique solution

$$f_\nu(0) = g_\nu\,, \quad \nu = 1, \ldots, n\,. \tag{5.27}$$

After substituting from (5.27) into (5.21) the relation (5.19) follows.

To demonstrate the remaining relation (5.20) it is necessary to consider the derivative of (5.3) at x = 0:

$$(5.28) \qquad x_\nu \rho_\nu + f'_\nu(0) + i\rho_\nu \sum_{\mu=1}^{n} f_\mu(0) - \rho_\nu \sum_{\mu=1}^{n} \frac{f'_\mu(0)}{x_\nu + x_\mu} = 0, \quad \nu = 1, \ldots, n \, .$$

Let the numbers h_ν be defined by the equation

$$(5.29) \qquad i\hat{t}(k) \sum_{\nu=1}^{n} \frac{h_\nu}{k+x_\nu} = ik\left\{r(k)-1\right\} + ik\hat{t}(k) - \hat{t}(k) \sum_{\nu=1}^{n} f_\nu(0) \, .$$

From (5.21), (5.27) and (4.2) it follows that

$$(5.30) \qquad \sum_{\nu=1}^{n} f_\nu(0) = \lim_{k \longrightarrow \infty} ik \left\{\frac{\hat{t}(k)-1}{\hat{t}(k)}\right\} .$$

Relation (5.30) is consistent with (5.29) as can be seen by allowing k to approach ∞ in (5.29) Since the poles on the left side of (5.29) are the same as those on the right, (5.29) is entirely consistent. The numbers h_ν are then given by

$$(5.31) \qquad h_\nu = \lim_{k \longrightarrow -x_\nu} (k+x_\nu)\left\{\frac{kr(k)+k\hat{t}(k)-k}{\hat{t}(k)} + i \sum_{\mu=1}^{n} f_\mu(0)\right\} = \frac{-x_\nu \left\{r(-x_\nu)-1\right\}}{\hat{t}-\nu}, \nu = 1, \ldots, n \, .$$

Also, from (5.29):

$$(5.32) \qquad \sum_{\mu=1}^{n} \frac{h_\mu}{x_\nu + x_\mu} = x_\nu \frac{\left\{r(x_\nu)+\hat{t}(x_\nu)-1\right\}}{\hat{t}(x_\nu)} + i \sum_{\mu=1}^{n} f_\mu(0), \quad \nu = 1, \ldots, n \, .$$

After substituting from (5.32) and (5.31) into the expression

$$(5.33) \qquad \rho_\nu \sum_{\mu=1}^{n} \frac{h_\mu}{x_\nu + x_\mu} \quad h_\nu$$

there results

$$\frac{\rho_\nu \varkappa_\nu \{r(\varkappa_\nu)+\hat{t}(\varkappa_\nu)-1\}}{\hat{t}(\varkappa_\nu)} + i\rho_\nu \sum_{\mu=1}^{n} f_\mu(0) + \frac{\varkappa_\nu \{r(-\varkappa_\nu)-1\}}{t_{-\nu}}$$

$$(5.34)\quad \begin{aligned} &= \lim_{k \to \varkappa_\nu} (k-\varkappa_\nu)\rho(k)k\left\{\frac{r(k)+\hat{t}(k)-1}{\hat{t}(k)}\right\} + i\rho_\nu \sum_{\mu=1}^{n} f_\mu(0) - \lim_{k \to \varkappa_\nu} (k-\varkappa_\nu)k\left\{\frac{r(-k)}{\hat{t}(-k)} - \frac{1}{\hat{t}(k)}\right\} \\ &= \lim_{k \to \varkappa_\nu} (k-\varkappa_\nu)k\left\{-\frac{r(k)r(-k)}{\hat{t}(-k)} + \rho(k) + \frac{r(-k)}{\hat{t}(-k)}\right\} \\ &+ i\rho_\nu \sum_{\mu=1}^{n} f_\mu(0) - \lim_{k \to \varkappa_\nu} (k-\varkappa_\nu)k\left\{\frac{r(-k)}{\hat{t}(-k)} - \frac{1}{\hat{t}(-k)}\right\} = \varkappa_\nu\rho_\nu + i\rho_\nu \sum_{\mu=1}^{n} f_\mu(0), \quad \nu = 1, \ldots, n, \end{aligned}$$

using the relations (4.20) of the lemma. The identity of the expressions (5.33) and (5.34) means that the numbers h_ν when substituted for the $f_\nu'(0)$ satisfy the system (5.28). Since (5.28) has a unique solution it follows that

$$(5.35)\qquad f_\nu'(0) = h_\nu\,, \quad \nu = 1, \ldots, n\,.$$

From (5.35) and (5.29) it follows that the relation (5.20) is satisfied. The remarks made concerning (5.19) and (5.20) show that

$$(5.36)\qquad t(k) = \hat{t}(k)\,.$$

The fact that the scattering matrix is unitary implies then that

$$(5.37)\qquad \rho(k) = \hat{\rho}(k)\,,$$

and Theorem 1 is proved.

It is now possible to obtain a closed form expression for the function $V(x)$ corresponding to $r(k)$. This result is contained in

Theorem 2: Given a reflection coefficient $r(k)$ of the form (4.1) subject to the general conditions on the scattering matrix elements and the special condition (4.2). The corresponding function $V(x)$ of (1.3) is then given by

$$V(x) = -\frac{d^2}{dx^2}\log\Delta(x), \qquad x < 0, \tag{5.38}$$

where $\Delta(x)$ is the determinant of the matrix whose elements have the form

$$\delta_{\nu\mu} + \frac{-\rho_\nu e^{i(x_\nu+x_\mu)x}}{x_\nu+x_\mu}, \qquad \nu, \mu = 1, \ldots, n. \tag{5.39}$$

Here the quantity $\delta_{\mu\nu}$ is the Kronecker delta

$$\delta_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu \end{cases}, \tag{5.40}$$

and the x_ν are roots in the upper half plane of the equation

$$1 - r(k)r(-k) = 0. \tag{5.41}$$

Proof: If the solutions $f_\nu(x)$ of the system of equations (5.3) are represented by Cramer's rule as ratios of determinants and are then substituted into (5.2) with t set equal to x, it can be seen easily that

$$\sum_{\mu=1}^{n} e^{-ix_\mu x} f_\mu(x) = -\frac{d\log\Delta(-x)}{dx}, \qquad x < 0. \tag{5.42}$$

It follows at once from (3.17) that

$$W(x) = -2\frac{d^2\log\Delta(-x)}{dx^2}, \qquad x < 0, \tag{5.43}$$

and from the definition (5.12) of $W(x)$ the theorem follows.

Finally, it should be noted that two linearly independent solutions of the differential equation (1.3) are given by

$$u(x,\pm k) = e^{\pm ikx} + ie^{\pm ikx} \sum_{\nu=1}^{n} \frac{f_\nu(-x)e^{i\varkappa_\nu x}}{\varkappa_\nu \pm k} . \tag{5.44}$$

6. Conclusion

For r(k) a reflection coefficient at x = 0 which is a rational function of k, the transmission coefficient, t(k), exists, is regular on the real axis and in the upper half plane and has no zeros on the real axis or in the upper half plane. The expressions for the scattering matrix elements t(k) and $\rho(k)$ in terms of r(k) are given in Section 4 in (4.19) and (4.18) when r(k) is given by (4.1). The expression for the corresponding V(x) is given in Section 5 in (5.38). The closing paragraphs of Section 5 following the proof of the theorem provide the necessary description for understanding (5.38). Two linearly independent solutions of the original differential equation (1.3) are given in (5.44).

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*The original method is actually an outgrowth of that of I.M. Gelfand and B.M. Levitan, 'On the determination of a differential equation by its spectral function', American Math. Translations, Ser. 2, 1, 253 (1955).

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